BRIEF COMMUNICATIONS

DRAG COEFFICIENTS FOR THE MOVEMENT
OF RIGID SPHERES THROUGH
LIQUID-FILLED CYLINDRICAL PORES

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Equivalent pore analysis has been used to describe lipid-insoluble solute permeation of capillary endothelial walls (1–4), red blood cell and axonal plasma membranes (5–9), and the nuclear envelope of eukaryotic cells (10, 11) in terms of movement through water-filled pores. It has also been used to predict the porosities of artificial membranes from permeability data (12–15). The validity of the approach is supported by (*a*) electron micrographs of capillary endothelia (16), nuclear envelopes (17), and artificial membranes (14, 15) which reveal transmembrane openings with approximately the dimensions determined by equivalent pore analysis of tracer permeation; (*b*) the ability of the analysis to reconcile markedly different permeabilities with a simple model (10); and (*c*) consistency with description of membrane properties by irreversible thermodynamics (18–20).

Mathematical treatment is simplest for ideal systems with (*i*) uncharged, cylindrical membrane pores of identical dimensions, (*ii*), rigid and spherical uncharged solute penetrants, and (*iii*) low Reynolds number solvents (21–24). Permeation in such systems may be viewed as a flux through uniform pores, each with an "effective" or "restricted" cross-sectional area (A_{eff}) which is equal to the true pore area (A_0) reduced by two factors: steric hindrance to pore entry, and the resistance or drag caused by the pore walls. Such restricted movement through membrane pores can be expressed by

$$A_{eff}/A_0 = \text{steric hindrance factor/wall drag factor.} \tag{1}$$

Since A_{eff} can be determined from permeability data, calculation of A_0 is possible when the steric hindrance and wall drag factors are known.

For systems with no net solvent flow, the steric hindrance factor can be obtained from the geometric condition that to enter a pore (radius *r*) a particle (radius *a*) can-

not strike the pore edge (1, 25); that is, only particles whose centers pass through the central circular area with radius $r-a$ enter. The factor is quantitatively expressed by

$$\pi(r-a)^2/\pi r^2 = (1-a/r)^2. \tag{2}$$

In systems with solvent flow through the pores due to osmotic or other pressure differentials, the steric hindrance factor is more complex. The correction required has been derived by Ferry (26), who showed the steric hindrance term for such filtration systems to be

$$(1-a/r)^2 \cdot [2-(1-a/r)^2]. \tag{3}$$

Stokes' Law for viscous drag on a spherical particle with radius a and velocity U in an unbounded fluid with viscosity η and velocity V, is

$$\text{Drag}_{\text{unb}} = 6\pi\eta a\,(U\text{-}V). \tag{4}$$

The comparable expression for the drag on a particle within a cylindrical pore is

$$\text{Drag}_p = 6\pi\eta a\,[U\kappa_1(a/r, \rho) - V\kappa_2(a/r, \rho)], \tag{5}$$

where $\kappa_1(a/r, \rho)$ and $\kappa_2(a/r, \rho)$ are the component drag coefficients which weight the contributions of the particle and fluid velocities, respectively. κ_1 and κ_2 are functions not only of a/r, but also of ρ, the radial distance of the particle center from the central pore axis. A drag factor, K, can be defined as the ratio of the drag on a particle within the pore to the drag in an unbounded medium, and is given by

$$K = (U\kappa_1 - V\kappa_2)/U\text{-}V. \tag{6}$$

For pure diffusional systems ($V = 0$) the drag factor equals κ_1, and Eq. 1 becomes

$$A_{\text{eff}}/A_0 = (1-a/r)^2/\kappa_1. \tag{7}$$

However, in systems with bulk solvent flow ($V \neq 0$), not only must one use the steric hindrance factor provided by Ferry (formula 3), but the contribution of κ_2 to the drag must be considered as well, and

$$A_{\text{eff}}/A_0 = (1-a/r)^2 \cdot [2-(1-a/r)^2]/K, \tag{8}$$

where *both* κ_1 and κ_2 contribute to the value of K through Eq. 6 (15, 27).[1]

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[1] Many biological applications of equivalent pore theory to date have ignored κ_2 and erroneously applied the equation $A_{\text{eff}}/A_0 = (1-a/r)^2 \cdot [2-(1-a/r)^2]/\kappa_1$ to systems with solvent flow (see ref. 15 for discussion). The magnitudes of κ_1 and κ_2 are similar, and the larger V is with respect to U the greater is the error associated with use of this equation.

TABLE I

$\kappa_1(a/r, 0)$ AND $\kappa_2(a/r, 0)$

Computed using 14 equations of the infinite set, as described in text. Values of Haberman and Sayre (31) in parentheses.

a/r	κ_1	κ_2	a/r	κ_1	κ_2
0.00	1.00000 (1.000)	1.00000 (1.000)	0.48	5.34141	4.53292
0.02	1.04393	1.04365	0.50	5.95938 (5.970)	4.98360 (5.004)
0.04	1.09178	1.09062	0.52	6.68043	5.50116
0.06	1.14397	1.14122	0.54	7.52686	6.09927
0.08	1.20096	1.19584	0.56	8.52705	6.79476
0.10	1.26330 (1.263)	1.25488 (1.255)	0.58	9.71752	7.60918
0.12	1.33159	1.31881	0.60	11.14580 (11.135)	8.57025 (8.651)
0.14	1.40654	1.38816	0.62	12.87453	9.71417
0.16	1.48892	1.46351	0.64	14.98751	11.08878
0.18	1.57966	1.54554	0.66	17.59862	12.75849
0.20	1.67980 (1.680)	1.63500 (1.635)	0.68	20.86550	14.81147
0.22	1.79054	1.73276	0.70	25.01092 (24.955)	17.37105 (17.671)
0.24	1.91328	1.83981	0.72	30.35733	20.61390
0.26	2.04963	1.95729	0.74	37.38467	24.80033
0.28	2.20150	2.08651	0.76	46.83132	30.32686
0.30	2.37109 (2.371)	2.22898 (2.231)	0.78	59.87967	37.82235
0.32	2.56100	2.38646	0.80	78.51925 (73.555)	48.33522 (47.301)
0.34	2.77430	2.56102	0.82	106.31670	63.72853
0.36	3.01464	2.75506	0.84	150.22684	87.60633
0.38	3.28635	2.97143	0.86	225.51931	127.82576
0.40	3.59464 (3.596)	3.21351 (3.218)	0.88	372.41035	204.96024
0.42	3.94578	3.48533	0.90	737.25652	393.56585
0.44	4.34741	3.79174			
0.46	4.80880	4.13856			

Both empirical and theoretical treatments have provided approximate solutions for $\kappa_1(a/r, \rho)$ and $\kappa_2(a/r, \rho)$ for particles moving in cylindrical tubes (22, 28–34). By limiting consideration to axisymmetric flow ($\rho = 0$), Haberman and Sayre (31) provided rigorous theoretical solutions for $\kappa_1(a/r, 0)$ and $\kappa_2(a/r, 0)$ for rigid spheres. Use of such center-line approximations, instead of coefficients formulated by radially averaging $\kappa_1(a/r, \rho)$ and $\kappa_2(a/r, \rho)$ over the entire pore area, introduces only small errors (15, 27, 35, 36); and Haberman and Sayre's solutions fit very well the available experimental results over the range $0 < a/r < 0.8$ (31, 33). Their determinations (done in 1958), requiring laborious computations, were tabulated only for solutions of $\kappa_1(a/r, 0)$ and $\kappa_2(a/r, 0)$ with a/r values of 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, and 0.8. These values were confirmed by Wang and Skalak (32).[2]

Investigators (e.g., ref. 19) have had to interpolate the values of $\kappa_1(a/r, 0)$ and $\kappa_2(a/r, 0)$ for values of a/r other than those above. Because these functions are not linear, accurate interpolation is difficult. Alternatively, one can use approximation formulae provided by Haberman and Sayre (31), but these are inaccurate for $a/r > 0.6$. We recently (10) needed values of $\kappa_1(a/r, 0)$ not available in the literature and, following the method of Haberman and Sayre (31), applied computer techniques to calculate the values of $\kappa_1(a/r, 0)$ and $\kappa_2(a/r, 0)$ over the range $0 < a/r \leq 0.90$ in 0.02 unit intervals (Table I).

[2]Chen and Skalak (34) extended the analysis to quantitate the effects of ellipsoidal penetrant shapes and the interactions between more than one penetrant particle present within the pore simultaneously.

Haberman and Sayre's solution for axisymmetric motion of a rigid sphere in still liquid inside a cylindrical tube is given in terms of an infinite set of infinite linear algebraic equations for the Stokes' stream function coefficients. They showed $\kappa_1(a/r, 0)$ to be directly related to one coefficient of this equation set (their Eq. 60) and, by using the first eight equations of the set, solved for the coefficient, and hence $\kappa_1(a/r, 0)$. They also applied the method to a system with moving fluid and a stationary sphere within the pore, generating a similar set of equations and calculating $\kappa_2(a/r, 0)$. The larger the a/r value for which $\kappa_1(a/r, 0)$ and $\kappa_2(a/r, 0)$ are computed, the greater is the number of equations of the infinite set required for the computations to converge to their asymptotic values. The extent of the calculations limited Haberman and Sayre to use of the first eight equations.

In our computations, an IBM 360 computer was used to obtain the necessary Bessel function integrals[3] and to solve for $\kappa_1(a/r, 0)$ and $\kappa_2(a/r, 0)$ using not only the first 8 equations of Haberman and Sayre, but also the first 10, 12, and 14 equations of the infinite set.[4] We found that the 8-equation systems provide solutions convergent (to the fifth decimal place) for $a/r < 0.56$, the 10-equation system for $a/r < 0.66$, and the 12-equation system for $a/r < 0.76$. Convergence is good up to $a/r = 0.90$, where the 12-equation and 14-equation solutions still agree to the nearest whole integer. Our values, computed from the 14-equation system, are given in Table I.

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[3] Bessel functions calculated using the BESK and IO subroutines (37).

[4] Set of simultaneous equations solved using a double precision version of the SIMQ subroutine (37).

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